A minimum of four terms in the solution form was necessary to predict accurately the divergence velocities as a function of the length-to-radius ratio ℓ/a. Divergence velocities varied widely before approaching a limit as the number of terms in the assumed solution was increased. This was particularly true for $\ell/a > 10$ and for higher-order modes ($n \geq 3$). A study of the effect of the thickness-to-radius ratio on critical divergence velocities showed that higher-order circumferential modes are associated with shorter and/or thinner shells. The lowest critical divergence velocity, for long shells ($\ell/a \geq 40$), is associated with the beam-type mode ($n = 1$).

Critical divergence velocity as a function of lamina orientation, for a given circumferential mode, is shown in Fig. 3 for $\ell/a = 10$ and $h/a = 0.01$. The present analysis was repeated for the isotropic case and compared with the results given in Ref. 7. Good agreement was noted.

Conclusions

Natural frequencies of fiber-reinforced shells decrease with increasing fluid velocity, as do isotropic shells, until static divergence occurs. Natural frequency is a function of the length-to-radius ratio, the radius-to-thickness ratio, the circumferential mode number, and the orientation of the lamina. Beam-type theory was found to be suitable for long shells, but in any other case shell theory must be used. Qualitatively, the behavior of anisotropic shells appears to be much the same as that of isotropic shells. However, it is reasonable to expect that certain lamina stacking sequences will result in considerably different numerical values of divergence velocity as a function of lamina orientation.

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Stability of Inviscid Shear Flow over Flexible Membranes

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Introduction

THE shear flow over compliant surfaces is physically interesting and its linear stability has been treated exhaustively in the viscous case.[1,2] In this Note we consider the inviscid limit and give several results for self-excited disturbances. For various flow parameters these include bounds on unstable eigenvalues, sufficient conditions for stability, and explicit dispersion relations.

Analysis

We orient along the x axis the velocity $U(y) + u(x,y,t)$ and along the y axis the velocity $v(x,y,t)$ with t being time. Consider a flexible membrane whose deviation from equilibrium $y = 0$ satisfies $y = n(x,t)$ and assume a rigid wall placed along $y = -H < 0$ across which $v = 0$. If $u,v \ll U$, the use of $u = \psi_y$, $v = -\psi_x$, and $\psi(x,y,t) = \phi(y)\exp\, ik(x-ct)$ leads to the linear Rayleigh equation:

$$(U-c)(\phi'' - k^2\phi) - U''\phi = 0 \tag{1}$$

where k is a specified wavenumber and $c = c_r + ic_i$ is the complex eigenvalue. At the wall

$$\phi(-H) = 0 \tag{2}$$

Next, the disturbance pressure acting on the membrane satisfies $p(y=0) = \rho g n - T n_{xx} + sn + mn_{tt} + mdn_t$. Here we assume a fluid density ρ, a gravitational acceleration g, a membrane tension T, a spring constant s, a lineal mass density m, and a damping constant d. Assuming $p = \hat{p}(y)\exp\, ik(x-ct)$ and $n = a\exp\, ik(x-ct)$ leads to $\hat{p}(0) = (\rho g + Tk^2 + s - mk^2c^2 - imdkc)a$, while the kinematic condition $v = n_t + Un_x$ leads to $\phi(0) = -(U(0) - c)a$. Hence,

$$\frac{\phi'(0)}{\phi(0)} = \frac{U'(0)}{U(0)-c} + \frac{\rho g + Tk^2 + s - mk^2c^2 - imdkc}{\rho[U(0)-c]^2} \tag{3}$$

Equations (1-3) complete the inviscid stability formulation. Some general consequences will now be discussed.

First Howard's[3] semicircle theorem is extended to handle inhomogeneous boundary conditions. Euqation (1) and the definitions $\phi(y) = (c-U)F(y)$ and $W = U - c$ imply that $(W^2F')' - k^2W^2F = 0$. Multiplication by the complex conjugate F^* and integration over $(-H,0)$ leads to

$$\int_{-H}^{0} W^2\{|F'|^2 + k^2|F|^2\}\mathrm{d}y = F^*(W^2F')|_{-H}^{0} = G|F_0|^2$$

where we have set $\rho G = \rho(G_r + iG_i) = \rho g + Tk^2 + s - mk^2c^2 - imdkc$, taken Eqs. (3) and (2) in the form $W^2F' = GF$ and $F(-H) = 0$, and written $F_0 = F(0)$. Now define $Q = |F'|^2 +$

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$k^2 |F|^2 > 0$ so that real and imaginary parts give

$$\int_{-H}^{0} [(U-c_r)^2 - c_i^2] Q\mathrm{d}y = G_r |F_0|^2 \tag{4}$$

$$-2c_i \int_{-H}^{0} (U-c_r) Q\mathrm{d}y = G_i |F_0|^2 \tag{5}$$

Note that $G_r = g + Tk^2/\rho + s/\rho - mk^2(c_r^2 - c_i^2)/\rho + mdkc_i/\rho$ and $G_i = -2mk^2 c_r c_i/\rho - mdkc_r/\rho$. Expansion of Eq. (4) using Eq. (5) leads to

$$\int_{-H}^{0} U^2 Q\mathrm{d}y = \left(G_r - \frac{c_r}{c_i} G_i\right) |F_0|^2 + (c_r^2 + c_i^2) \int_{-H}^{0} Q\mathrm{d}y \tag{6}$$

assuming $c_i > 0$. Let A and B be the minimum and maximum of $U(y)$ in $(-H, 0)$. Then the identity

$$0 \geq \int_{-H}^{0} (U-A)(U-B) Q\mathrm{d}y = \int_{-H}^{0} U^2 Q\mathrm{d}y - (A+B) \int_{-H}^{0} UQ\mathrm{d}y + AB \int_{-H}^{0} Q\mathrm{d}y$$

on using Eqs. (5) and (6) yields

$$0 \geq [(c_r^2 + c_i^2) - (A+B)c_r + AB]\left[\frac{mdk}{\rho c_i} |F_0|^2 + \frac{mk^2}{\rho} |F_0|^2 + \int_{-H}^{0} Q\mathrm{d}y\right] + \left[g + \frac{s}{\rho} + \frac{k^2}{\rho}(T - mAB) + \frac{mdkc_r(A+B)}{2\rho c_i} - \frac{mdkAB}{\rho c_i}\right] |F_0|^2 \tag{7}$$

Several cases are distinguished. For nondissipative membranes ($d=0$), note

$$0 \geq [(c_r^2 + c_i^2) - (A+B)c_r + AB]\left[\frac{mk^2}{\rho} |F_0|^2 + \int_{-H}^{0} Q\mathrm{d}y\right] + [g + \frac{s}{\rho} + \frac{k^2}{\rho}(T - mAB)] |F_0|^2$$

If $g + s/\rho + k^2(T - mAB)/\rho \geq 0$, it follows that $c_r^2 + c_i^2 - (A+B)c_r + AB \leq 0$ or

$$[c_r - \tfrac{1}{2}(A+B)]^2 + c_i^2 \leq [\tfrac{1}{2}(A+B)]^2 - AB = [\tfrac{1}{2}(B-A)]^2 \tag{8}$$

Thus, the complex wave velocity c for any unstable mode must lie inside the semicircle in the upper half of the complex c-plane which has the range of U for diameter. Alternatively, for massless membranes, $m=0$, and Eq. (7) again simplifies considerably. If $g + s/\rho + Tk^2/\rho > 0$, Eq. (8) again holds with the same consequences. Results for more general membrane parameters have not been obtained and probably require the use of different inequalities.

We now obtain sufficient conditions for stability for light membranes. The formula used in proving Rayleigh's theorem, that is,

$$\int_{-H}^{0} \{ |\phi'|^2 + k^2 |\phi|^2 \} \mathrm{d}y + \int_{-H}^{0} \frac{U'' |\phi|^2}{U-c} \mathrm{d}y = \phi' \phi^* |_{-H}^{0}$$

is evaluated using Eqs. (2) and (3). The resulting imaginary part is:

$$c_i \left\{ \int_{-H}^{0} \frac{U'' |\phi|^2}{|U-c|^2} \mathrm{d}y - \left[\frac{2(U-c_r)G_r + \{(U-c_r)^2 - c_i^2\}(G_i/c_i)}{|U-c|^4} + \frac{U'}{|U-c|^2}\right] |\phi|^2_{y=0} \right\} = 0$$

Setting $m=0$ leads to $G_i = 0$ and $G_r = g + Tk^2/\rho + s/\rho > 0$. Hence,

$$c_i \left\{ \int_{-H}^{0} \frac{U'' |\phi|^2}{|U-c|^2} \mathrm{d}y - \left[\frac{2(U-c_r)[g + (Tk^2/\rho) + s/\rho]}{|U-c|^4} + \frac{U'}{|U-c|^2}\right] |\phi|^2_{y=0} \right\} = 0 \tag{9}$$

Now suppose that $U''(y) < 0$ and $U'(0) \geq 0$ and that $c_i \neq 0$. Equation (9) then requires that $U(0) = \max U < c_r$. However, if this is satisfied, c_i must vanish according to the semicircle theorem. This contradicts the supposition $c_i \neq 0$. Hence, c_i must be zero. Thus, if $U'' < 0$ throughout and $U'(0) \geq 0$, the flow is stable. Similarly, if $U'' > 0$ throughout and $U'(0) \leq 0$, the flow is also stable (these results do not apply to constant U flows since $U' = U'' = 0$). Sufficiency conditions for more general membrane parameters have not been obtained.

Specific Examples

Explicit solutions for straight-line velocity profiles are easy to obtain and we thus examine more general classes of problems. However, we restrict ourselves to those yielding simple closed-form solutions. For long waves we assume $|\phi''| \gg k^2 |\phi|$ in Eq. (1). The Rayleigh equation reduces to $(U-c)\phi'' - U''\phi = 0$. Defining $\Phi = \phi/(U-c)$ and using $\phi(-H) = 0$ lead to an integrable equation whose solution is of the form

$$\phi(y) = C_0 (U-c) \int_{-H}^{y} (U-c)^{-2} \mathrm{d}y$$

From this, the expression for $\phi'(y)/\phi(y)$ can be calcualted. Evaluation at $y=0$ and comparison with Eq. (3) yields the dispersion relation

$$\int_{-H}^{0} \frac{\mathrm{d}y}{(U-c)^2} = 1 \Big/ \left[g + \frac{Tk^2}{\rho} + \frac{s}{\rho} - \frac{mk^2c^2}{\rho} - \frac{imk\mathrm{d}c}{\rho}\right]$$

The corresponding eigenfunction is:

$$\phi(y) = -\frac{a}{\rho}[\rho g + Tk^2 + s - mk^2c^2 - imk\mathrm{d}c] \times [U-c] \int_{-H}^{y} \frac{\mathrm{d}y}{(U-c)^2}$$

where C_0 was determined by evaluating $\phi(y)$ at $y=0$ and using the dispersion and surface kinematic conditions. The foregoing solutions hold for long waves only; the integrals are evaluated by assuming $c_i > 0$.

Closed-form solutions are also possible for a restricted class of shear flows. Assuming that $U'(y)$ is small near the membrane, an approximate solution to Eq. (1) is $\phi(y) = -a(U-c)e^{ky}$. Solutions of this form have been considered in various hydrodynamic stability studies. The foregoing solution satisfies the kinematic condition $\phi(0) \cong -a(U(0) - c)$ as well as $\phi(-\infty) = 0$. The approximation holds whenever either of $U'/k(U-c)$ or kU'/U'' is small. The first forbids critical points and the second is a restriction to long waves. The surface condition leads to $\rho(-(U(0)-c)\phi'(0)+\phi(0)U'(0))=(\rho g+Tk^2+s-mk^2c^2-imdkc)a$. Then, evaluation with the assumed solution produces the approximate dispersion relation $\rho(U_0-c)^2k=\rho g+Tk^2+s-mk^2c^2-imdkc$ where $U_0=U(0)$. Thus unstable c_i's are easily determined for all parameters.

Finally, consider an interesting example for the Couette flow $U(y)=\beta y+\gamma$. In this case $U''=0$ and if $(U-c)$ is nonzero in $(-H,0)$, the solution ϕ to $\phi''-k^2\phi=0$ satisfying Eq. (2) is proportional to $\tanh kH \cosh ky+\sinh ky$. Substitution in Eq. (3) leads to the dispersion relation

$$\frac{\phi'(0)}{\phi(0)}=\frac{(\gamma-c)\beta\rho+\rho g+Tk^2+s-mk^2c^2-imdkc}{\rho(\gamma-c)^2}$$

$$=\frac{k}{\tanh kH}$$

Now this is equivalently $\phi(0)=R^{-1}\phi'(0)$ where R measures an effective rigidity. In the rigid wall limit we find that $\phi(0)=0$ and hence ϕ vanishes identically; the basic flow, therefore, has no eigensolutions of the discrete c-spectrum. Thus we conclude here that it is the allowance of wall flexibility that gives rise to wavelike perturbations.

Discussion and Concluding Remarks

The foregoing results for the inviscid shear flow stability over compliant surfaces were obtained to supplement the more detailed viscous studies of Benjamin[1] and Landahl[2] using the Orr-Sommerfeld equation. Our work focuses essentially on explicitly obtainable eigenvalue bounds, sufficiency conditions, and dispersion relations for the Benjamin-Landahl membrane model, although, for somewhat restrictive parameter ranges.

While the results reported here are not directly applicable to the analysis of finite length plates (panel flutter) per se as contrasted with infinite length membranes, the general methods may be used in conjunction with alternative structural models and with only minor modification. The basic problem, of course, is one for fluid and solid interaction, and a substantial body of more general work appears in the aeroelastic literature. Noteworthy among these are some significant theoretical models for shear flows over flexible boundaries, for example, those pursued by Dowell and his collaborators.[4-6] These studies allow for both finite plate dimensions as well as fluid compressibility; their numerical results, moreover, compare favorably with the experiments of Muhlstein et al.[7,8] The analytical approaches adopted here and in Refs. 4-6 are more or less equivalent. The simpler interaction model examined here, though, leads to simple closed-form results, and these may be useful in various applications.

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Use of Matched Pressure Initial Conditions for Predicting Low-Altitude Rocket Plume Radiation

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Nomenclature

A_{max} = plume maximum cross-sectional area, $A_{max}=9\pi(FD)^{1/2}/16q_\infty$

A = nozzle area

C_D = plume drag coefficient: $C_D=D/q_\infty A_{max}$

C_F = nozzle thrust coefficient:

$$C_F=\left\{\frac{2\gamma_e^2}{\gamma_e-1}\left(\frac{2}{\gamma_e+1}\right)^{(\gamma_e+1)/(\gamma_e-1)}\times\left[1-\left(\frac{p_e}{p_c}\right)^{(\gamma_e-1)/\gamma_e}\right]\right\}^{1/2}+\frac{p_e-p_\infty}{p_c}\frac{A_e}{A^*}$$

$$C_{F_{max}}=\left\{\frac{2\gamma_e^2}{\gamma_e-1}\left(\frac{2}{\gamma_e+1}\right)^{(\gamma_e+1)/(\gamma_e-1)}\times\left[1-\left(\frac{p_\infty}{p_c}\right)^{(\gamma_e-1)/\gamma_e}\right]\right\}^{1/2}$$

D = plume drag

F = missile thrust

M = Mach number

p = pressure

q = dynamic pressure

r = plume or nozzle radius

T = temperature

v = velocity

γ = specific heat ratio

Subscripts

e = nozzle exit

∞ = ambient

c = combustion chamber

Superscript

$*$ = nozzle throat

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